A Finsler Elastic Minimal Path Model for Perceptual Grouping

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Finsler Elastic Metric: Minimal path problem [3] is posed on a bounded domain Ω and a metric $\mathcal{F}$ (potentially asymmetric) at each point $\mathbf{x} \in \Omega$. Let $\mathcal{A}$ denote the collection of Lipschitz paths $\Gamma : [0, 1] \to \Omega$, and the path length ℓ can be measured through the metric $\mathcal{F}$ as:

$$\ell(\Gamma) = \int_0^1 \mathcal{F}\big(\Gamma(t), \dot{\Gamma}(t)\big)\, dt, \tag{1}$$

where $\dot{\Gamma}(t) = \frac{d}{dt}\Gamma(t)$. The minimal action map $\mathcal{U}$ from an initial source points $\mathbf{s}$, is the minimal energy of any path joining $\mathbf{x} \in \Omega$ to $\mathbf{s}$:

$$\mathcal{U}(\mathbf{x}) := \min\big\{\ell(\Gamma); \Gamma \in \mathcal{A}, \Gamma(1) = \mathbf{x}, \Gamma(0) = \mathbf{s}\big\}. \tag{2}$$

The minimal action map $\mathcal{U}$ is the unique viscosity solution to an Eikonal PDE, defined in terms of the dual metric $\mathcal{F}^*$. The minimal action map $\mathcal{U}$ satisfies the Eikonal equation for each $\mathbf{x} \in \Omega$:

$$\begin{cases} \mathcal{F}^*\big(\mathbf{x}, -\nabla \mathcal{U}(\mathbf{x})\big) = 1, \\ \mathcal{U}(\mathbf{s}) = 0, \end{cases} \quad \text{where } \mathcal{F}^*(\mathbf{x}, \vec{u}) = \sup_{\vec{v} \neq \mathbf{0}} \frac{\langle \vec{u}, \vec{v}\rangle}{\mathcal{F}(\mathbf{x}, \vec{v})}. \tag{3}$$

The metric $\mathcal{F}$ considered in this paper combine a symmetric part, defined in terms of a symmetric positive tensor field $\mathcal{M}$, and an asymmetric part involving a vector field $\vec{\omega}$:

$$\mathcal{F}(\mathbf{x}, \vec{u}) = \sqrt{\langle \vec{u}, \mathcal{M}(\mathbf{x})\, \vec{u}\rangle} - \langle \vec{\omega}(\mathbf{x}), \vec{u}\rangle. \tag{4}$$

We require $\langle \vec{\omega}(\mathbf{x}), \mathcal{M}^{-1}(x)\, \vec{\omega}(x)\rangle < 1$ to ensure the metric positivity.

Following Mumford [6], the weighted Euler's elastica curves, minimizing the following bending energy:

$$\mathcal{L}(\Gamma) = \int_0^L \frac{1}{\Phi\big(\Gamma(s)\big)} \Big(1 + \alpha \kappa^2(s)\Big)\, ds \tag{5}$$

L denotes the classical curve length, s is the arc-length parameter. κ is the curvature and Φ is a velocity function.

The first step is to cast the elastica energy (5) in the form of path length with respect to a degenerate Finsler metric. For that purpose, let $\mathbb{S}^1 = [0, 2\pi)$ be the space of angles with periodic boundary conditions. For each θ, let $\vec{v}_\theta = (\cos\theta, \sin\theta)$ be the corresponding unit vector. It is known that

$$\frac{d}{dt}\left(\frac{\dot{\Gamma}(t)}{\|\dot{\Gamma}(t)\|}\right) = \kappa(t)\|\dot{\Gamma}(t)\| \left(\frac{\dot{\Gamma}(t)}{\|\dot{\Gamma}(t)\|}\right)^{\perp}.$$

D efining Euclidean arc-length by $ds = \|\Gamma'(t)\| dt$, one has

$$\begin{aligned} \int_0^L (1 + \alpha\kappa^2(s))\, ds &= \int_0^1 \left(\|\dot{\Gamma}(t)\| + \frac{\alpha\,|\dot{\theta}(t)|^2}{\|\dot{\Gamma}(t)\|}\right) dt \\ &= \int_0^1 \mathcal{F}^\infty\big(\gamma(t), \dot{\gamma}(t)\big)\, dt, \end{aligned} \tag{6}$$

where we define the Finsler metric $\mathcal{F}^\infty$ as

$$\mathcal{F}\infty(\bar{\mathbf{x}}, \bar{\mathbf{u}}) := \begin{cases} \|\mathbf{u}\| + \frac{\alpha\,|\nu|^2}{\|\mathbf{u}\|}, & \text{if } \mathbf{u} \propto \vec{v}_\theta, \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

for any orientation lifted point $\bar{\mathbf{x}} = (\mathbf{x}, \theta) \in \bar{\Omega} = \Omega \times \mathbb{S}^1$, any vector $\bar{\mathbf{u}} = (\mathbf{u}, \nu) \in \mathbb{R}^2 \times \mathbb{R}$ in the tangent space, and where $\propto$ denotes positive collinearity. The Finsler metric $\mathcal{F}^\infty$ defined in (7) is too singular to apply the numerical algorithm such as Fast Marching method [5] to compute $\mathcal{U}$. Hence we introduce a family of orientation lifted Finsler metrics over the lifting domain $\bar{\Omega}$, depending on a penalization parameter $\lambda \gg 1$ as follows:

$$\mathcal{F}^\lambda(\bar{\mathbf{x}}, \bar{\mathbf{u}}) := \sqrt{\lambda^2\|\mathbf{u}\|^2 + 2\alpha\lambda|\nu|^2} - (\lambda - 1)\langle \vec{v}_\theta, \mathbf{u}\rangle, \tag{8}$$

As $\lambda \to \infty$ one has:

$$\mathcal{F}^\lambda(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \mathcal{O}\left(\frac{1}{\lambda}\right) + \|\mathbf{u}\| + \frac{\alpha|\nu|^2}{\|\mathbf{u}\|} - (\lambda - 1)\Big(\|\mathbf{u}\| - \langle \vec{v}_\theta, \mathbf{u}\rangle\Big),$$

which tends to $\mathcal{F}^\infty$ as $\lambda \to \infty$. In order to apply $\mathcal{F}^\lambda$ to image analysis, we incorporate the image data dependent velocity function Φ to $\mathcal{F}^\lambda$ to obtain the weighted Finsler elastic metric $\mathcal{J}$:

$$\mathcal{J}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = \frac{\sqrt{\lambda^2\|\mathbf{u}\|^2 + 2\alpha\lambda|\nu|^2}}{\Phi(\bar{\mathbf{x}})} - (\lambda - 1)\frac{\langle \vec{v}_\theta, \mathbf{u}\rangle}{\Phi(\bar{\mathbf{x}})}. \tag{9}$$

where $\alpha > 0$ is a constant. Φ is defined over the orientation lifted domain and should be large along the expected image features such as boundaries or curves. We calculate Φ by the steerable edge detectorinvoking high order Gaussian kernel [4]. The minimal action map associated to metric $\mathcal{J}$, denoted by $\mathcal{W}_\mathbf{s}$ can be efficiently computed by the Fast Marching method [5] with $\mathbf{s}$ denoting the initial source point.

Perceptual Grouping: Perceptual grouping is relevant to the task of curve reconstruction and completion [2]. The class of geodesic distance based perceptual grouping models was firstly introduced by [2] using the saddle points to identify the pairs of points which have to be linked by minimal paths among the set of key points. Later on, this idea was improved by [1] by the anisotropic Riemannian metric instead of the isotropic version adopted in [2]. In this paper, we focus on the perceptual grouping problem of finding closed contours formed by piecewise smooth minimal paths with positions in the set $\mathcal{H}_1 \subseteq \mathcal{H}$, where

$$\mathcal{H} := \{\mathbf{x}_i \in \Omega \subset \mathbb{R}^2, i = 1, 2, ..., m; m \geq 2\},$$

is a collection of physical points provided by user and the orientation lifting of $\mathcal{H}$ is defined as

$$\begin{aligned} \mathcal{D} := \Big\{ &\bar{\mathbf{x}}_i = (\mathbf{x}_i, \theta_i), \bar{\mathbf{x}}_i^\dagger = \big(\mathbf{x}_i, \mathrm{mod}(\theta_i + \pi, 2\pi)\big); \\ &i = 1, 2, ..., m, \text{ and } \theta_i \in [0, 2\pi)\Big\}, \end{aligned} \tag{10}$$

This grouping problem can be converted to the task of finding closed curves consisting of curvature penalized minimal paths with the lifting endpoints in $\mathcal{D}_1 \subseteq \mathcal{D}$, which can be done by searching the pairs of lifting points from $\mathcal{D}$ that have to be joined by minimal paths. The orientation lifting set $\mathcal{D}$ is constructed by manually assigning orientations to each physical position in $\mathcal{H}$ respectively, as expressed in (10). For each physical position $\mathbf{x} \in \mathcal{D}$, there exists two orientation lifting points: $\bar{\mathbf{x}}$ and $\bar{\mathbf{x}}^\dagger$. The set $\mathcal{D}_1$ can be identified through the geodesic distance $\mathcal{W}$ with respect to the Finsler elastic metric $\mathcal{J}$ (9).

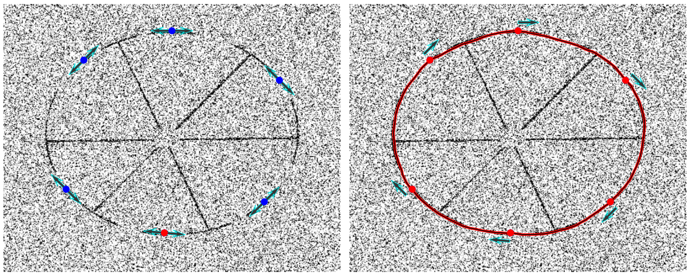
We firstly specify the first physical point $\mathbf{x}_1$ to initialize the algorithm. The corresponding orientation lifting points of $\mathbf{x}_1$, denoting by $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_1^\dagger$, can be automatically chosen from $\mathcal{D}$ and will be removed from $\mathcal{D}$. Then the closest vertices $\bar{\mathbf{z}}^*, \bar{\mathbf{z}}_\dagger^*$ corresponding to $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_1^\dagger$ can be respectively identified by

$$\bar{\mathbf{z}}^* := \arg\min_{\bar{\mathbf{z}} \in \mathcal{D}} \mathcal{W}_{\bar{\mathbf{x}}_1}(\bar{\mathbf{z}}), \quad \text{and} \quad \bar{\mathbf{z}}_\dagger^* := \arg\min_{\bar{\mathbf{z}} \in \mathcal{D}} \mathcal{W}_{\bar{\mathbf{x}}_1^\dagger}(\bar{\mathbf{z}}). \tag{11}$$

With those definitions, the first and second vertice $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2$ are chosen simultaneously by the following criterion:

$$(\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2) := \begin{cases} (\bar{\mathbf{x}}_1, \bar{\mathbf{z}}^*), \text{ if } \mathcal{W}_{\bar{\mathbf{x}}_1}(\bar{\mathbf{z}}^*) < \mathcal{W}_{\bar{\mathbf{x}}_1^\dagger}(\mathbf{z}_\dagger^*), \\ (\bar{\mathbf{x}}_1^\dagger, \bar{\mathbf{z}}_\dagger^*), \text{ otherwise.} \end{cases} \tag{12}$$

As a result, the first pair of vertice $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2 \in \mathcal{D}$ is computed using (12) and the geodesic $\mathcal{C}_{\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2}$ is recovered.



(a) (b)

Figure 1: Perceptual grouping results. (**a**) Initialization: red and blue dots are physical positions, in which the red dot is the selected initial position. (**b**) Grouping results. Arrows indicate the tangents for each physical positions denoted by red dots.

Once the first pair of vertice $\{\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2\}$ is found, we add $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2$ to $\mathcal{D}_1$, remove $\bar{\mathbf{q}}_2$ from $\mathcal{D}$ and compensate $\bar{\mathbf{q}}_1$ to $\mathcal{D}$. Next vertice $\bar{\mathbf{q}}_3$ is found by

$$\bar{\mathbf{q}}_3 := \arg\min_{\bar{\mathbf{z}} \in \mathcal{D}} \mathcal{W}_{\bar{\mathbf{q}}_2}(\bar{\mathbf{z}}). \tag{13}$$

We remove $\bar{\mathbf{q}}_3$ from $\mathcal{D}$. Again, $\bar{\mathbf{q}}_3$ is added to $\mathcal{D}_1$ and the geodesic $\mathcal{C}_{\bar{\mathbf{q}}_2, \bar{\mathbf{q}}_3}$ between $\bar{\mathbf{q}}_2$ and $\bar{\mathbf{q}}_3$ is recovered.

We stop the perceptual grouping method once the vertex $\bar{\mathbf{q}}_1$ are identified by using (13). In other words, once $\bar{\mathbf{q}}_1$ can minimize the geodesic distance with respect to initial source point $\bar{\mathbf{q}}_i$ where $\bar{\mathbf{q}}_i$ is the latest vertex of $\mathcal{D}_1$, we stop the algorithm.

The proposed perceptual grouping method can be extended to search n grouping subsets $\mathcal{D}_i$ ($i = 1, 2 \ldots n$) with constraint $\cap_{i=1}^n \mathcal{D}_i = \varnothing$. When $\mathcal{D}_1$ is found, we remove the first vertex $\bar{\mathbf{q}}_1$ from $\mathcal{D}$. Then using the same search procedure as $\mathcal{D}_1$, one can easily find $\mathcal{D}_2$ from $\mathcal{D}$. Note that the removed vertice in the previous searching steps will not be compensated to $\mathcal{D}$.

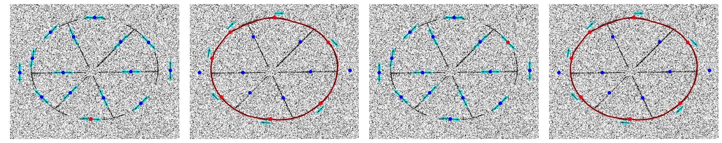
Numerical Experiments: The perceptual grouping result on a synthetic noisy image is shown in Fig. 1. In Fig. 1(a), we demonstrate the original image consisting of a set of edges. Red and blue dots with arrows are the lifting points provided by user as the initialization, where the red dot is the selected initial physical position. (b) shows the perceptual grouping results by the proposed method. The identified lifting points in the set $\mathcal{D}_1$ are denoted by red dots with arrows. Red curves linking the lifting points indicate the expected closed curves.

Fig. 2 illustrates the capacity of the proposed curvature penalized minimal path based perceptual grouping method to deal with the edge map with spurious lifting points. In Fig. 2(a) and (c), different initializations are shown. Red dots are the selected initial physical positions. (b) and (d) are the grouping results. Red curves linking the lifting points indicate the expected closed curves.

The proposed perceptual grouping method can detect more than one closed curves by only specifying the number of expected closed curves. In Fig. 3, three closed curves are detected. In Fig. 3, row 1 shows different initializations with red dots denoting the initial physical position. Row 2 illustrates the first detected closed curve indicated by red curves. Red dots with arrows denote the selected lifting points in $\mathcal{D}_1$. Row 3 illustrates the second detected closed curve indicated by orange curves. Orange dots with arrows illustrate the lifting points in $\mathcal{D}_2$. The initial physical positions are selected **randomly** after $\mathcal{D}_1$ was detected. Row 4 demonstrates the third closed curve using the similar procedure to the detection of $\mathcal{D}_2$. We show the final closed curve detection results in Row 5 which means that the final results do not depend on the first point selection, i.e., with any initial physical position, our algorithm can obtain the expected results.

Remak: In this paper, the orientation lifting points collection is given manually at this time. Actually, those physical points can be automatically using method similar to [2] and the orientation for each physical position can be detected using for instance the corresponding rotated gradient vector of the edges.

Conclusion: We present a perceptual grouping method using Finsler elastic minimal path model. This method can group edges into a closed contour



(a) (b) (c) (d)

Figure 2: Perceptual grouping results. (**a**) and (**c**) Initializations: Red dots are the selected initial position. (**b**) and (**d**) Perceptual grouping results for initializations in (**a**) and (**c**) respectively. Red dots and the corresponding arrows are the lifting points chosen to form a closed curve.

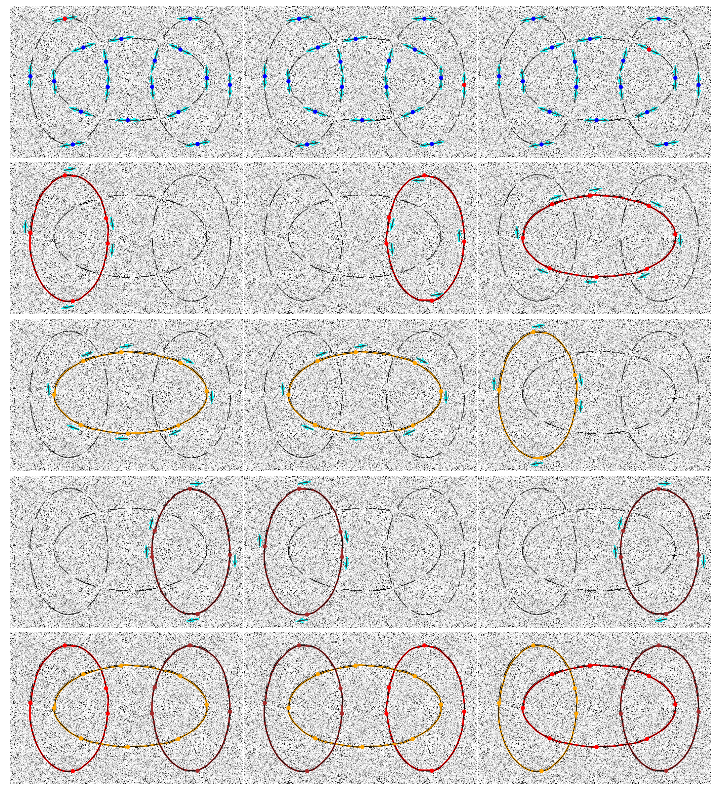


Figure 3: Perceptual grouping results by identifying the given lifting points to three groups. **Row 1** initializations. Red dots are the selected initial positions. **Rows 2-4** intermediate grouping results for the corresponding initializations. **Row 5** final grouping results.

through a set of user-provided orientation-lifted points in term of curvature penalized geodesic distance. Experimental results on noisy images show that our method indeed can obtain the desired grouping results.

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